s-LECTURE HALL PARTITIONS, SELF-RECIPROCAL POLYNOMIALS, AND GORENSTEIN CONES

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Abstract. In 1997, Bousquet–Mélou and Eriksson initiated the study of *lecture hall partitions*, a fascinating family of partitions that yield a finite version of Euler's celebrated odd/distinct partition theorem. In subsequent work on **s**-lecture hall partitions, they considered the *self-reciprocal property* for various associated generating functions, with the goal of characterizing those sequences **s** that give rise to generating functions of the form $((1-q^{e_1})(1-q^{e_2})\cdots(1-q^{e_n}))^{-1}$.

We continue this line of investigation, connecting their work to the more general context of Gorenstein cones. We focus on the Gorenstein condition for **s**-lecture hall cones when **s** is a positive integer sequence generated by a second-order homogeneous linear recurrence with initial values 0 and 1. Among such sequences **s**, we prove that the n-dimensional **s**-lecture hall cone is Gorenstein for all $n \geq 1$ if and only if **s** is an ℓ-sequence. One consequence is that among such sequences **s**, unless **s** is an ℓ-sequence, the generating function for the **s**-lecture hall partitions can have the form $((1-q^{e_1})(1-q^{e_2})\cdots(1-q^{e_n}))^{-1}$ for at most finitely many n.

We also apply the results to establish several conjectures by Pensyl and Savage regarding the symmetry of h^*-vectors for **s**-lecture hall polytopes. We end with open questions and directions for further research.

Contents

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1. Overview

We will use polyhedral geometry to make progress on some open questions in the theory of *lecture hall partitions*.

1.1. **Lecture hall partitions and generating functions.** For a sequence $\mathbf{s} = \{s_i\}_{i\geq 1}$ of positive integers, let $\mathcal{L}_n^{(\mathbf{s})}$ denote the set of all $\mathbf{s}$-*lecture hall partitions* of length n,

$$\mathcal{L}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}. \tag{1}$$

In [3] and [4], Bousquet–Mélou and Eriksson consider the generating function for lecture hall partitions,

$$f_n^{(\mathbf{s})}(q) \;=\; \sum_{\lambda \in \mathcal{L}_n^{(\mathbf{s})}} q^{|\lambda|},$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$. In [3], they show that for the sequence $\mathbf{s} = (1, 2, \ldots, n)$ (this sequence gave rise to the name *lecture hall partition* since one interpret the parts as admissible heights of seats in a lecture hall), this generating function has the form

$$f_n^{(1,2,\ldots,n)}(q) \;=\; \sum_{\lambda \in \mathcal{L}_n^{(1,2,\ldots,n)}} q^{|\lambda|} \;=\; \prod_{i=1}^{n} \frac{1}{1 - q^{2i-1}}\,. \tag{2}$$

In partition theory, this result is notable not only because of the surprisingly simple generating function, but also because it is an entirely new finite form of *Euler's Partition Theorem*, which asserts that the number of partitions of an integer M into distinct parts is the same as the number of partitions of M into odd parts. Note that the right-hand side of (2) is the generating function for partitions (of any integer) into parts from the set $\{1, 3, \ldots, 2n-1\}$, which approaches the set of all odd parts as $n \to \infty$. Correspondingly, on the left-hand side, as $n \to \infty$, the set $\mathcal{L}_n^{(1,2,\ldots,n)}$ becomes the set of partitions into distinct parts.

In [4], Bousquet–Mélou and Eriksson show that a similar phenomenon occurs for a more general class of sequences which they refer to as (k,ℓ)-sequences. Given positive integers k and ℓ, the (k,ℓ)-*sequence* $\mathbf{a} = (a_i)_{i=0}^{\infty}$ is defined by $a_0 = 0$, $a_1 = 1$, and for $i \geq 1$,

$$a_{2i} = \ell a_{2i-1} - a_{2i-2} \qquad \text{and} \qquad a_{2i+1} = k a_{2i} - a_{2i-1}\,.$$

The following generalization of (2) was proved in [4].

Theorem 1.1 (Bousquet–Mélou and Eriksson [4])**.** *For positive integers $k, \ell \geq 2$, let $\mathbf{a}$ be the (k,ℓ)-sequence and let $\mathbf{b}$ be the corresponding (ℓ,k)-sequence. Then*

$$f_n^{(\mathbf{a})}(q) = \sum_{\lambda \in \mathcal{L}_n^{(\mathbf{a})}} q^{|\lambda|} = \prod_{i=0}^{\lceil n/2 \rceil - 1} \frac{1}{1 - q^{a_{2n-i} + a_{2n-i-1}}} \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{1 - q^{b_{2n+1-i} + b_{2n-i}}}\,. \tag{3}$$

Note that if $k = \ell$, then $\mathbf{a} = \mathbf{b}$ and the generating function of the theorem becomes

$$\prod_{i=1}^{n} \frac{1}{1 - q^{a_i + a_{i-1}}}\,, \tag{4}$$

which simplifies to (2) when $k = \ell = 2$. When $k = \ell$, $\mathbf{a}$ is called an *ℓ-sequence* [11]. Thus the ℓ-sequences are defined for a positive integer $\ell > 0$ by

$$a_{i+1} = \ell a_i - a_{i-1}\,, \tag{5}$$

with initial conditions $a_1 = 1$, $a_0 = 0$.

To appreciate the significance of (2) and (3), note that for a general sequence $\mathbf{s}$ of positive integers, the $\mathbf{s}$-lecture hall partitions have a generating function of the form

$$f_n^{(\mathbf{s})}(q) = \frac{H(q)}{\prod_{i=1}^{n}(1 - q^{s_i + \cdots + s_n})}\,, \tag{6}$$

where $H(q)$ is a polynomial with integer coefficients satisfying $H(1) = \prod_{i=1}^{n} s_i$. (See, e.g., [9, Thm. 5], where $H(q)$ is given a combinatorial interpretation.) So, in the case of (k, ℓ)-sequences, the numerator $H(q)$ factors and is completely cancelled by the denominator.

It is natural to consider if there might be other sequences $\mathbf{s}$ for which $f_n^{(\mathbf{s})}$ would have the form

$$f_n^{(\mathbf{s})}(q) = \prod_{i=1}^{n} \frac{1}{(1 - q^{e_i})} \tag{7}$$

for some positive integers $e_1, \ldots, e_n$. In [4], Bousquet–Mélou and Eriksson investigate sequences $\mathbf{s}$ having this property. Their approach (and ours) is to study self-reciprocal generating functions.

1.2. **Lecture hall partitions and self-reciprocal generating functions.** A rational function $r(q)$ is *self-reciprocal* if there exists a nonnegative integer k such that

$$r(1/q) = \pm q^k\, r(q)\,.$$

Note that if $f_n^{(\mathbf{s})}(q)$ is of the form (7), then $f_n^{(\mathbf{s})}(q)$ is self-reciprocal, since

$$f_n^{(\mathbf{s})}\left(\frac{1}{q}\right) = \frac{(-1)^n q^{e_1 + e_2 + \cdots + e_n}}{\prod_{i=1}^{n}(1 - q^{e_i})}\,.$$

This led Bousquet–Mélou and Eriksson to investigate in [4] the relationship between the condition defining the (k, ℓ)-sequences and the property that the generating function $f_n^{(\mathbf{s})}(q)$ is self-reciprocal. Bousquet–Mélou and Eriksson define a sequence $\mathbf{s}$ to be *polynomic* if $f_n^{(\mathbf{s})}(q)$ is the multiplicative inverse of a polynomial; hence, the (k, ℓ)-sequences are polynomic. They conjecture that in some sense all polynomic sequences arise from (k, ℓ)-sequences, and they prove the following partial characterization.

Theorem 1.2 (Bousquet–Mélou and Eriksson [4])**.** *If $\mathbf{s}$ is a non-decreasing sequence of positive integers with the property that $\gcd(s_i, s_{i-1}) = 1$ for $1 < i \le n$, then $f_n^{(\mathbf{s})}(q)$ is self-reciprocal if and only if $s_i + s_{i-2}$ is a multiple of s_i, for $3 \le i \le n$, and $s_2 + 1$ is a multiple of s_1.*

Theorem 1.2 can be applied to show that the (k,ℓ)-sequences have self-reciprocal generating functions. It also implies that the generating function for the $(1,3,5,7)$-lecture hall partitions is self-reciprocal. However, the sequence $\mathbf{s} = (1,3,5,7)$ is not polynomic: it was shown in [4] that

$$f_4^{(1,3,5,7)}(q) = \frac{H(q)}{(1-q^7)(1-q^{12})(1-q^{15})(1-q^{16})} = \frac{1-q+q^3-q^4+q^5-q^7+q^8}{(1-q)^2(1-q^{12})(1-q^{16})}\,,$$

where

$$\begin{aligned} H(q) = {} & q^{28}+q^{27}+q^{26}+2\,q^{25}+2\,q^{24}+3\,q^{23}+4\,q^{22}+3\,q^{21}+4\,q^{20} \\ & +5\,q^{19}+5\,q^{18}+6\,q^{17}+6\,q^{16}+6\,q^{15}+q^{14}+6\,q^{13}+6\,q^{12}+6\,q^{11} \\ & +5\,q^{10}+5\,q^{9}+4\,q^{8}+3\,q^{7}+4\,q^{6}+3\,q^{5}+2\,q^{4}+2\,q^{3}+q^{2}+q+1\,. \end{aligned} \tag{8}$$

Theorem 1.2 can also be applied to get negative results. For example, for the first five terms of the Fibonacci sequence $\mathbf{s} = (1,1,2,3,5)$, the generating function $f_5^{(\mathbf{s})}(q)$ is not self-reciprocal since $5+2$ is not a multiple of 3. As a consequence, $f_5^{(\mathbf{s})}(q)$ cannot have the form (7).

But many sequences are not covered by Theorem 1.2. For example $s = (1,3,2,1,3,2)$ is not monotone; $\mathbf{s} = (1,3,18,81,405,1944)$ does not satisfy $\gcd(s_i, s_{i+1}) = 1$. We will show in Section 2 that both sequences give rise to s-lecture hall partitions with self-reciprocal generating functions. One of our main contributions is the following result, which is implied by Theorem 1.4 (in Section 2.1) and Theorem 3.2 (in Section 3).

Theorem 1.3. *Let b and ℓ be integers satisfying $0 < |b| \le \ell$. Let $\mathbf{s}$ be defined by the recurrence*

$$s_n = \ell s_{n-1} + b s_{n-2}\,, \tag{9}$$

with initial conditions $s_1 = 1$, $s_0 = 0$. The lecture hall generating function $f_n^{(\mathbf{s})}(q)$ is self-reciprocal for all $n \ge 0$ if and only if $b = -1$. If $b \ne -1$, there is an integer $n_0 = n_0(b,\ell)$ so that for all $n \ge n_0$, $f_n^{(\mathbf{s})}(q)$ is not self-reciprocal and, consequently, $f_n^{(\mathbf{s})}(q)$ cannot have the form $((1-q^{e_1})(1-q^{e_2})\cdots(1-q^{e_n}))^{-1}$.

The ℓ-sequences have special significance in partition theory, giving rise to an ℓ-version of Euler's partition theorem [4, 11]. But it has not been clear whether other sequences might behave similarly, or what other properties might distinguish ℓ-sequences. Theorem 1.3 demonstrates that ℓ-sequences play a characterizing role in the study of $\mathbf{s}$-lecture hall partitions.

The proof of Theorem 1.3, and of our extension of Theorem 1.2 to more general sequences, involves properties of Gorenstein cones. The Gorenstein condition provides a framework for relating self-reciprocity of generating functions to arithmetic properties of the integer points in a rational pointed cone, a framework that has been established using the theory of normal semigroup algebras in combinatorial commutative algebra.

1.3. **Gorenstein lecture hall cones.** In partition theory, the defining constraints for $\mathcal{L}_n^{(\mathbf{s})}$ are unusual, dealing with the *ratio* of consecutive parts of the partition λ, rather than more typical constraints, such as the difference of consecutive parts, or the set of allowable parts. However, within the theory of lattice-point enumeration in rational polyhedral cones, the lecture-hall constraints are just special cases of general linear constraints bounding a region that contains the points of

interest. This is not a new observation; in [4, Section 5], Bousquet–Mélou and Eriksson point out that the study of lecture hall partitions falls naturally within the theory of linear homogeneous diophantine systems of inequalities. However, this observation provides a wide variety of algebraic and combinatorial tools for applications, including the Gorenstein property.

Recall that a *polyhedral cone* in $\mathbb{R}^n$ is the solution set to a finite collection of linear inequalities $Ax \geq 0$ for some real matrix A. The cone is *rational* if A has rational entries, it is *simple* if it can be described by n inequalities, and it is *pointed* if it does not contain a line. A pointed rational cone $\mathcal{C} \subset \mathbb{R}^n$ is *Gorenstein* if there exists an integer point $\mathbf{c}$ in the interior $\mathcal{C}^\circ$ of $\mathcal{C}$ such that $\mathcal{C}^\circ \cap \mathbb{Z}^n = \mathbf{c} + (\mathcal{C} \cap \mathbb{Z}^n)$. We call $\mathbf{c}$ a *Gorenstein point* of $\mathcal{C}$.

For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, the $\mathbf{s}$-*lecture hall cone* $\mathcal{C}_n^{(\mathbf{s})}$ is defined by

$$\mathcal{C}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}.$$

Observe that $\mathcal{C}_n^{(\mathbf{s})}$ is a pointed rational cone and $\mathcal{L}_n^{(\mathbf{s})} = \mathcal{C}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$. The key property of Gorenstein cones is the following.

Theorem 1.4. *For a sequence* $\mathbf{s}$ *of positive integers, the following are equivalent:*

- $\mathcal{C}_n^{(\mathbf{s})}$ *is Gorenstein;*
- $f_n^{(\mathbf{s})}(q) = \sum_{\lambda \in \mathcal{L}_n^{(\mathbf{s})}} q^{|\lambda|}$ *is self-reciprocal;*
- $H(q)$ *is a palindromic polynomial.*

Here we write, as in (6), $f_n^{(\mathbf{s})}(q) = \frac{H(q)}{\prod_{i=1}^n (1-q^{s_i+\cdots+s_n})}$.

A proof of Theorem 1.4 is given in Section 2.1. In Section 2.2, we develop tools for checking the Gorenstein condition for $\mathcal{C}_n^{(\mathbf{s})}$. In Section 2.3, we use these tools to generalize Theorem 1.2 and construct sequences that give rise to Gorenstein lecture hall cones.

Theorem 1.2 leaves open the possibility that sequences of the form (9), other than ℓ-sequences, could have self-reciprocal generating functions. For example, when $\ell = 3$ and $b = 9$, $\mathbf{s} = (1, 3, 18, 81, 405, 1944, \ldots)$. As will be shown in Section 2, $f_6^{(\mathbf{s})}(q)$ is self-reciprocal, but $f_7^{(\mathbf{s})}(q)$ is not. In Section 3, we prove Theorem 3.2, which states that the ℓ-sequences are unique among second order linear recurrences in the following sense: $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein for all $n \geq 1$ if and only if $\mathbf{s}$ is an ℓ-sequence. The proof relies on the tools developed in Section 2.2 together with a delicate analysis of how $\gcd(s_i, s_{i+1})$ is related to b and ℓ.

Another consequence of our main results involves the h^*-vector of certain polytopes associated with the $\mathbf{s}$-lecture hall partitions. For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, the *rational lecture hall polytope* $\mathcal{R}_n^{(\mathbf{s})}$ is defined by

$$\mathcal{R}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathcal{C}_n^{(\mathbf{s})} : \lambda_n \leq 1 \right\}.$$

Note that $\mathcal{R}_n^{(\mathbf{s})}$ is a rational simplex, i.e., the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. In Section 4, we define the h^*-vector of $\mathcal{R}_n^{(\mathbf{s})}$ and show it is symmetric if and only if $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein. This settles some conjectures from [8].

2. Gorenstein s-lecture hall cones

2.1. **Background.** Throughout this work, we will rely heavily on the following special case of a result due to Stanley.

Theorem 2.1 (Stanley [12]). *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be an n-dimensional pointed rational cone. Let w be a weight function defined on the integer lattice points in $\mathcal{C}$ satisfying*

- *$w(u)$ is a nonnegative integer,*
- *$w(u+v) = w(u) + w(v)$,*
- *$w(u) = 0$ implies $u = 0$, and*
- *for any nonnegative integer m, $w^{-1}(m)$ is a finite set.*

Let $f(q) = \sum_{u \in \mathcal{C} \cap \mathbb{Z}^n} q^{w(u)}$. Then $\mathcal{C}$ is Gorenstein if and only if $f(q) = (-1)^n q^b f(1/q)$ for some nonnegative integer b.

Before sketching a proof of Theorem 2.1, we use it to prove Theorem 1.4.

Proof of Theorem 1.4. The weight function $w(\lambda) = |\lambda| = \sum_i \lambda_i$ satisfies the conditions of Theorem 2.1, so the first two items are equivalent. Let $e_i = s_i + \cdots + s_n$. For the last two items, $f_n^{(\mathbf{s})}(q)$ being self-reciprocal means that

$$\frac{H(q)}{\prod_{i=1}^n (1-q^{e_i})} = (-1)^n q^b \frac{H(1/q)}{\prod_{i=1}^n (1-(1/q)^{e_i})} = \frac{(-1)^n q^b H(1/q)}{(-1)^n q^{\sum_i e_i} \prod_{i=1}^n (1-q^{e_i})} = \frac{q^{b-\sum_i e_i} H(1/q)}{\prod_{i=1}^n (1-q^{e_i})},$$

that is,

$$H(q) = q^{b-\sum_i e_i} H(1/q).$$

This implies that $b = \deg(H) + \sum_i e_i$ and that $H(q)$ has symmetric coefficients. □

In the example following Theorem 1.2, we observed that the $(1,3,5,7)$-lecture hall partitions have a self-reciprocal generating function. Note that its corresponding $H(q)$ shown in (8) is palindromic.

The definition of a Gorenstein cone is motivated by a connection between pointed rational cones and normal Gorenstein semigroup algebras that underlies Theorem 2.1; we include here a sketch of the proof of Theorem 2.1. This sketch is meant to be a helpful outline of the proof for those unfamiliar with the relevant techniques in combinatorial commutative algebra. The details of the proof involve the Cohen–Macaulay property of normal semigroup algebras. Throughout we refer to the text [5], where references to original sources may be found. We also recommend Stanley's paper [12].

Sketch of proof of Theorem 2.1. The integer points in the cone $\mathcal{C}$ form a semigroup, and from this the semigroup algebra $k[\mathcal{C}]$ is constructed. This algebra admits a finite set of generators arising from the Hilbert basis of $\mathcal{C}$ (one may instead choose to use the ray generators and the integer point in the fundamental parallelepiped of $\mathcal{C}$). Corollary 6.3.6 in [5] states Hochster's result that $k[\mathcal{C}]$ is Cohen–Macaulay for an admissible grading, which our weight function provides. Corollary 6.3.8 in [5] states Stanley's result that the cone $\mathcal{C}$ is Gorenstein if and only if $k[\mathcal{C}]$ is a Gorenstein algebra; Gorenstein algebras form a special subclass of Cohen–Macaulay algebras.

For a positively-graded, finitely-generated Cohen–Macaulay algebra A, one can associate a Hilbert series $f(q)$, which corresponds to our $f(q)$ above. Corollary 4.4.6 in [5] states Stanley's result that a positively-graded, finitely-generated Cohen–Macaulay algebra A that is an integral domain is Gorenstein if and only if $f(q)$ is self-reciprocal, meaning that

$$f(q) = (-1)^{\dim(A)} q^b f\left(\frac{1}{q}\right),$$

for some integer b, where $\dim(A)$ is the Krull dimension of A.

Observe that [5, Corollary 6.3.6] implies that $k[\mathcal{C}]$ is finitely generated and Cohen–Macaulay, the structure of the semigroup given by the integer points in $\mathcal{C}$ guarantees that $k[C]$ is an integral domain, and our weight function w yields a positive grading. The result follows from the final observation that the Krull dimension of $k[\mathcal{C}]$ is given by the dimension of $\mathcal{C}$. $\square$

2.2. **Testing for the Gorenstein property.** The following lemma demonstrates how one can construct Gorenstein points for Gorenstein **s**-lecture hall cones.

Lemma 2.2. *Let $\mathcal{C} = \{\lambda \in \mathbb{R}^n : A\lambda \geq 0\}$ be a simple polyhedral cone, where A is a lower triangular matrix with positive entries on the diagonal. Denoting the rows of A as linear functionals $\alpha^1, \ldots, \alpha^n$ on $\mathbb{R}^n$, define a point $\mathbf{c} \in \mathcal{C}^\circ$ by the following algorithm. For $1 \leq i \leq n$, choose $c_i \in \mathbb{Z}$ minimal so that $\alpha^i(\mathbf{c}) = \alpha^i(c_1, \ldots, c_i) > 0$. (This choice is possible because A has positive entries on the diagonal.) If $\mathcal{C}$ is Gorenstein, then $\mathbf{c}$ is the unique Gorenstein point of $\mathcal{C}$.*

Proof. The point $\mathbf{c}$ lies in the interior of $\mathcal{C}$ because $\alpha^i(\mathbf{c}) > 0$ for $i = 1, \ldots, n$. Let $\mathcal{C}$ be Gorenstein, and let $\hat{\mathbf{c}}$ be a Gorenstein point. We prove the following property for all $j = 0, \ldots, n$ by induction: $\hat{c}_i = c_i$ for all $i \leq j$. For $j = 0$, nothing is to be shown. For $j > 0$, since $\mathbf{c} \in \hat{\mathbf{c}} + (\mathcal{C} \cap \mathbb{Z}^n)$, we find

$$\alpha^j(c_1, \ldots, c_{j-1}, c_j) \geq \alpha^j(\hat{c}_1, \ldots, \hat{c}_{j-1}, \hat{c}_j) = \alpha^j(c_1, \ldots, c_{j-1}, \hat{c}_j) > 0.$$

Thus $c_j \geq \hat{c}_j$, and, due to the minimal choice of c_j, actually $c_j = \hat{c}_j$. $\square$

The next lemma provides one of our main tools for checking the Gorenstein condition of an **s**-lecture hall cone.

Lemma 2.3. *Let the notation be as in Lemma 2.2 and assume that A is rational. For $j = 1, \ldots, n$, let the projected lattice $\alpha^j(\mathbb{Z}^n) \subset \mathbb{R}$ be generated by the number $q_j \in \mathbb{Q}_{>0}$, so $\alpha^j(\mathbb{Z}^n) = q_j\mathbb{Z}$. Then $\mathcal{C}$ is Gorenstein if and only if there exists $\mathbf{c} \in \mathbb{Z}^n$ such that $\alpha^j(\mathbf{c}) = q_j$ for all $j = 1, \ldots, n$.*

Proof. Let $\mathbf{b}^1, \ldots, \mathbf{b}^n \in \mathbb{Z}^n$ be the primitive generators of the rational cone $\mathcal{C}$, labeled such that we have the biorthogonality relation $\alpha^i(\mathbf{b}^j) = 0$ for $i \neq j$ and $\alpha^i(\mathbf{b}^i) > 0$. Without loss of generality, by scaling α^i, we can assume that $\alpha^i(\mathbf{b}^i) = 1$.

Assume $\mathcal{C}$ is Gorenstein, with Gorenstein point $\mathbf{c}$. Suppose that there exists an index j such that $\alpha^j(\mathbf{c}) \neq q_j$ (and so $\alpha^j(\mathbf{c}) > q_j$.) Because $\alpha^j(\mathbb{Z}^n) = q_j\mathbb{Z}$, there exists a point $\mathbf{z} \in \mathbb{Z}^n$ with $\alpha^j(\mathbf{z}) = q_j$. For $i \neq j$, let $\kappa_i = 1 + \max\{0, \lceil -\alpha^i(\mathbf{z}) \rceil\}$. Then $\hat{\mathbf{z}} := \mathbf{z} + \sum_{i \neq j} \kappa_i \mathbf{b}^i \in \mathcal{C}^\circ \cap \mathbb{Z}^n$ and $\alpha^j(\hat{\mathbf{z}}) = \alpha^j(\mathbf{z}) = q_j$. Since $\alpha^j(\hat{\mathbf{z}}) = q_j < \alpha^j(\mathbf{c})$, we find that $\hat{\mathbf{z}}$ does not lie in $\mathbf{c} + \mathcal{C}$, and thus $\mathcal{C}$ is not Gorenstein.

Now suppose that for some $\mathbf{c} \in \mathbb{Z}^n$, $\alpha^j(\mathbf{c}) = q_j$ for all $j = 1, \ldots, n$. Then $\mathbf{c} \in \mathcal{C}^\circ$. Let $\mathbf{z} \in \mathcal{C}^\circ \cap \mathbb{Z}^n$, so $\alpha^j(\mathbf{z}) \in q_j\mathbb{Z}_{>0}$ for all $j = 1, \ldots, n$. Then $\alpha^j(\mathbf{z} - \mathbf{c}) \geq 0$ for all $j = 1, \ldots, n$, and so $\mathbf{z} \in \mathbf{c} + \mathcal{C}$. Thus $\mathcal{C}$ is Gorenstein. □

Corollary 2.4. *For a sequence* $\mathbf{s}$, *the* $\mathbf{s}$*-lecture hall cone* $\mathcal{C}_n^{(\mathbf{s})}$ *is Gorenstein if and only there exists* $\mathbf{c} \in \mathbb{Z}^n$ *satisfying*

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_j, s_{j-1})$$

for $j > 1$, *with* $c_1 = 1$.

Proof. We use the characterization of Lemma 2.3. We have

$$\begin{aligned}
\alpha^1 &= (\tfrac{1}{s_1}, 0, \ldots, 0),\\
\alpha^2 &= (-\tfrac{1}{s_1}, \tfrac{1}{s_2}, 0, \ldots, 0)\\
\alpha^3 &= (0, -\tfrac{1}{s_2}, \tfrac{1}{s_3}, 0, \ldots, 0)\\
&\vdots\\
\alpha^n &= (0, \ldots, 0, -\tfrac{1}{s_{n-1}}, \tfrac{1}{s_n})\,.
\end{aligned}$$

So, $q_1 = 1/s_1$ and $q_j = 1/\operatorname{lcm}(s_{j-1}, s_j)$ for $2 \leq j < n$. By Lemma 2.3, $\mathcal{C}$ is Gorenstein if and only if there exists $\mathbf{c} \in \mathbb{Z}^n$ satisfying

$$\begin{aligned}
\alpha^1(\mathbf{c}) &\overset{!}{=} \frac{1}{s_1}\\
\alpha^2(\mathbf{c}) &\overset{!}{=} \frac{1}{\operatorname{lcm}(s_1, s_2)}\\
\alpha^3(\mathbf{c}) &\overset{!}{=} \frac{1}{\operatorname{lcm}(s_2, s_3)}\\
&\vdots\\
\alpha^n(\mathbf{c}) &\overset{!}{=} \frac{1}{\operatorname{lcm}(s_{n-1}, s_n)}\,.
\end{aligned}$$

Since $\alpha^1(\mathbf{c}) = c_1/s_1$ and $\alpha^1(\mathbf{c}) = -c_{j-1}/s_{j-1} + c_j/s_j$ for $2 \leq j < n$, this is equivalent to $c_1 = 1$ and, for $2 \leq j < n$,

$$c_j s_{j-1} - s_j c_{j-1} = \frac{s_{j-1} s_j}{\operatorname{lcm}(s_j, s_{j-1})} = \gcd(s_j, s_{j-1})\,.$$ □

In [4], Bousquet–Mélou and Eriksson proved a version of Corollary 2.4 for non-decreasing sequences of positive integers $\mathbf{s}$, using their proof of the generalized lecture hall theorem. Note in Corollary 2.4 that we do not need the condition that $\mathbf{s}$ is non-decreasing.

Corollary 2.4 gives a recurrence for c_j and therefore implies the following.

Corollary 2.5. *If for some* n *the sequence* $(c_1, \ldots, c_n)$ *satisfies the condition of Lemma 2.3, then* $(c_1, \ldots, c_j)$ *also satisfies it for all* $1 \leq j \leq n$. *Thus, if an* $\mathbf{s}$*-lecture hall cone fails to be Gorenstein in some fixed dimension, it fails for all higher dimensions as well.*

Example. Consider the sequence $\mathbf{s} = (1, 3, 18, 81, 405, 1944, 9477, \ldots)$ defined by the linear recurrence (9) with $\ell = 3$ and $b = 9$. The sequence of greatest common divisors $\gcd(s_i, s_{i-1})_{i\geq 1}$ is $(1, 1, 3, 9, 81, 81, 243, \ldots)$. Applying Corollary 2.4 to compute the Gorenstein point $\mathbf{c}$ of $\mathcal{C}_n^{(\mathbf{s})}$ gives $\mathbf{c} = (1, 4, 25, 113, 566, 2717, 26491/2, \ldots)$. Thus, by Corollaries 2.4 and 2.5, $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if $n \leq 6$.

The following lemma provides another method of checking the Gorenstein condition; while we do not use this in our present work, we include it, as it might prove useful for other applications.

Lemma 2.6. *Let $\mathcal{C} = \{\lambda \in \mathbb{R}^n : A\lambda \geq 0\}$ be a simple rational polyhedral cone. Denote the rows of A as linear functionals $\alpha^1, \ldots, \alpha^n$ on $\mathbb{R}^n$. For $j = 1, \ldots, n$, let the projected lattice $\alpha^j(\mathbb{Z}^n) \subset \mathbb{R}$ be generated by the number $q_j \in \mathbb{Q}_{>0}$, so $\alpha^j(\mathbb{Z}^n) = q_j\mathbb{Z}$. Define a point $\tilde{\mathbf{c}} \in \mathcal{C} \cap \mathbb{Q}^n$ by $\alpha^j(\tilde{\mathbf{c}}) = q_j$ for all $j = 1, \ldots, n$. Then $\mathcal{C}$ is Gorenstein if and only if $\tilde{\mathbf{c}} \in \mathbb{Z}^n$.*

2.3. **A construction for Gorenstein lecture hall cones.** In this section we apply Corollary 2.4 to show that there are many general sequences $\mathbf{s}$ that give rise to Gorenstein cones. This will slightly generalize Theorem 1.2.

Say that a sequence $\mathbf{s}$ of positive integers is $\mathbf{u}$-*generated* by a sequence $\mathbf{u}$ of positive integers if $s_2 = u_1 s_1 - 1$ and $s_{i+1} = u_i s_i - s_{i-1}$ for $i \geq 1$. For example, the (k, ℓ)-sequences are $\mathbf{u}$-generated by $\mathbf{u} = (\ell + 1, k, \ell, k, \ell, \ldots)$.

We will prove the following generalization of [4, Proposition 5.5]:

Theorem 2.7. *Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be a sequence of positive integers such that $\gcd(s_i, s_{i+1}) = 1$ for $1 \leq i < n$. Then $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if $\mathbf{s}$ is $\mathbf{u}$-generated by some sequence $\mathbf{u} = (u_1, u_2, \ldots, u_{n-1})$ of positive integers. When such a sequence exists, the Gorenstein point $\mathbf{c}$ for $\mathcal{C}_n^{(\mathbf{s})}$ is defined by $c_1 = 1$, $c_2 = u_1$, and for $2 \leq i < n$, $c_{i+1} = u_i c_i - c_{i-1}$.*

Consider the sequence $\mathbf{s} = (1, 3, 2, 1, 3, 2, 1, 3, 2, 1, \ldots)$ which is not monotone and therefore not covered by Theorem 1.2. It is $\mathbf{u}$-generated by $\mathbf{u} = (4, 1, 2, 5, 1, 2, 5, 1, 2, 5, \ldots)$, and therefore by Theorem 2.7, $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein and its Gorenstein point is $\mathbf{c} = (1, 4, 3, 2, 7, 5, 3, 10, 7, \ldots)$.

As another example, the "1 mod k" sequences

$$\mathbf{s} \ = \ (1,\ k+1,\ 2k+1,\ \ldots,\ (n-1)k+1)$$

were studied in [10], and in [8] it was conjecured that the h^*-vector of $\mathcal{R}_n^{(\mathbf{s})}$ is always symmetric. These 1 mod k sequences are $\mathbf{u}$-generated by $\mathbf{u} = (k + 2, 2, 2, 2, \ldots)$ and therefore, by Theorem 2.7, give rise to Gorenstein cones. In Section 4, we will show that this implies symmetry of the h^*-vector. The Gorenstein point is $(1, k + 2, 3k + 2, 5k + 1, \ldots)$.

Proof of Theorem 2.7. We adapt the method of [4]. By Corollary 2.4, $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if there exists $\mathbf{c} \in \mathbb{Z}^n$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_j, s_{j-1})$$

for $j > 1$, with $c_1 = 1$. Since $\gcd(s_i, s_{i+1}) = 1$, for $1 < j < n$,

$$c_j s_{j-1} - c_{j-1} s_j = 1 = c_{j+1} s_j - c_j s_{j+1}.$$

So, we can conclude that c_j and s_j are relatively prime and

$$c_j(s_{j-1} + s_j + 1) = s_j(c_{j+1} + c_{j-1}).$$

But then, since $\gcd(c_j, s_j) = 1$, $s_{j-1} + s_{j+1}$ must be a multiple of s_j, i.e., for some positive integer u_j,

$$s_{j+1} = u_j s_j - s_{j-1}.$$

Since $c_2 s_1 = s_2 c_1 + 1 = s_2 + 1$, setting $u_1 = s_2 + 1 = c_2$ ensures that $\mathbf{s}$ is $\mathbf{u}$-generated.

For the Gorenstein point, for $j > 1$, we have $c_1 = 1$, $c_2 = u_1$ and for $j > 1$,

$$s_j(c_{j+1} + c_{j-1}) = c_j(s_{j-1} + s_{j+1}) = c_j u_j s_j,$$

so $c_{j+1} + c_{j-1} = c_j u_j$, as claimed. □

3. The Gorenstein condition and second order linear recurrences

Assume that for a fixed $\ell \geq 0$ and $b \in \mathbb{Z}$, the sequence $\mathbf{s}$ is defined recursively through

$$s_0 := 0, \ s_1 := 1, \qquad \text{and} \qquad s_j := \ell s_{j-1} + b s_{j-2} \ \text{ for } \ j \geq 2\,. \tag{10}$$

If $b = -1$, $\mathbf{s}$ is an ℓ-sequence as noted in (5). In this section we will prove that among second order linear recurrences with initial conditions 0 and 1, only the ℓ-sequences give rise to Gorenstein lecture hall cones in every dimension n.

We will make use of the following proposition, which indicates that ℓ-sequences exhibit interesting arithmetic behavior. Specifically, Proposition 3.1 implies that ℓ-sequences are precisely those sequences that are as close as possible between weak and strict log-concavity. (A sequence (a_n) is *strictly log concave* if $a_n^2 > a_{n-1}a_{n+1}$ and *weakly log concave* if $a_n^2 \geq a_{n-1}a_{n+1}$.)

The following relation between terms of $\mathbf{s}$ is well known (see, e.g., [2, Identity 46]).

Proposition 3.1. *Let* $\mathbf{s}$ *be defined through* (10)*. Then for all* $0 \leq j \leq n$*,*

$$s_n^2 - s_{n+1}s_{n-1} = (-b)^{n-1}\,.$$

In particular, if $\mathbf{s}$ *an* ℓ*-sequence and* $j = n-1$*, then we have* $s_j^2 = s_{j-1}s_{j+1} + 1$*.*

Our main contribution is the following.

Theorem 3.2. *Let* b *and* ℓ *be integers satisfying* $0 < |b| \leq \ell$*. Let* $\mathbf{s} = (s_1, s_2, \ldots)$ *be defined by*

$$s_n = \ell s_{n-1} + b s_{n-2}\,,$$

with initial conditions $s_1 = 1$*,* $s_0 = 1$*. Then* $\mathcal{C}_n^{(\mathbf{s})}$ *is Gorenstein for all* $n \geq 0$ *if and only if* $b = -1$*. If* $b \neq -1$*, there exists* $n_0 = n_0(b, \ell)$ *such that* $\mathcal{C}_n^{(\mathbf{s})}$ *fails to be Gorenstein for all* $n \geq n_0$*.*

(The conditions on b and ℓ ensure that $s_i > 0$ for $i \geq 1$ and that the recurrence is of order two.)

Our analysis splits into two cases depending on whether b is negative (Case 1) or positive (Case 2). In Case 1, in order that $s_i > 0$ for $i > 0$, it must be that $-\ell < b < 0$. Both cases rely, ultimately, on Corollaries 2.4 and 2.5.

3.1. Case 1: $-\ell < b < 0$.

Proposition 3.3. *Let* $\mathbf{s}$ *be defined through* (10)*. Assume that* $-\ell < b < 0$*. Then for all* $n > 1$*,*

$$s_n > (-b)^{n-1}. \tag{11}$$

Proof. We use induction on n and the fact that $-b$ is positive and that $\ell \geq -b+1$. For the base case, $s_2 = \ell > (-b)$ and

$$s_3 \;=\; \ell^2 + b \;\geq\; (-b+1)^2 + b \;=\; (-b)^2 - b + 1 > (-b)^2.$$

Assume (11) is true for $n-1$ and $n-2$; then

$$\begin{aligned} s_n \;=\; \ell s_{n-1} + b s_{n-2} &> \ell(-b)^{n-2} + b(-b)^{n-3} \\ &\geq (-b+1)(-b)^{n-2} + b(-b)^{n-3} \\ &= (-b)^{n-1} + (-b)^{n-2} + b(-b)^{n-3} \\ &= (-b)^{n-1} + (-b)(-b)^{n-3} + b(-b)^{n-3} \\ &= (-b)^{n-1}\,. \end{aligned}$$

□

Proposition 3.4. *If* $\mathbf{s}$ *is a sequence given by* (10) *with* $-\ell < b < 0$*, then the unique possible Gorenstein point* $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ *for* $\mathcal{C}_n^{(\mathbf{s})}$ *has coordinates* $c_1 = 1$ *and*

$$c_j = s_j + s_{j-1} \qquad \text{for } j > 1\,. \tag{12}$$

Proof. We use the algorithm of Lemma 2.2. The choice for c_1 is clear, and we recursively compute the other coordinates by choosing a minimal c_j such that

$$\frac{s_{j-1} + s_{j-2}}{s_{j-1}} < \frac{c_j}{s_j}\,, \tag{13}$$

i.e., by Proposition 3.1,

$$s_j\,(s_{j-1} + s_{j-2}) = s_j s_{j-1} + s_{j-1}^2 - (-b)^{j-2} < s_{j-1} c_j\,,$$

i.e., since $s_{j-1} > 0$,

$$s_j + s_{j-1} - \frac{(-b)^{j-2}}{s_{j-1}} < c_j\,,$$

and with $0 < \frac{(-b)^{j-2}}{s_{j-1}} < 1$ (where the last inequality is Proposition 3.3). We have

$$c_j \geq s_j + s_{j-1}\,.$$

However, $s_j + s_{j-1}$ satisfies (13), and so we are done. □

The following proposition is our key tool to understanding the Gorenstein condition in Case 1.

Proposition 3.5. *Fix* (n, ℓ, b)*. The cone* $\mathcal{C}_n^{(\mathbf{s})}$ *is Gorenstein if and only if* $\gcd(s_{j-1}, s_j) = (-b)^{j-2}$ *for all* $j \leq n$*.*

Proof. Corollary 2.4 says that $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if its unique Gorenstein candidate from Lemma 2.2 satisfies

$$\gcd(s_{j-1}, s_j) = c_j s_{j-1} - c_{j-1} s_j \,.$$

Now use Propositions 3.4 and 3.1:

$$\gcd(s_{j-1}, s_j) = s_{j-1}^2 - s_{j-2} s_j = (-b)^{j-2}. \qquad \square$$

Theorem 3.6. *Let* $\mathbf{s}$ *be a sequence given by* (10) *with* $-\ell < b < 0$.

(i) *If* $b = -1$, *i.e.,* $\mathbf{s}$ *is an* ℓ*-sequence, then* $\mathcal{C}_n^{(\mathbf{s})}$ *is Gorenstein for all* n.
(ii) *If* $n \leq 2$, *then* $\mathcal{C}_n^{(\mathbf{s})}$ *is Gorenstein.*
(iii) *If* $n = 3$, *then* $\mathcal{C}_n^{(\mathbf{s})}$ *is Gorenstein if and only if* $b \mid \ell$.
(iv) *Otherwise, if* $b < -1$ *and* $n \geq 4$, *then* $\mathcal{C}_n^{(\mathbf{s})}$ *is not Gorenstein.*

Proof. We check the characterization given by Proposition 3.5.

(i) We have $\gcd(s_1, s_2) = \gcd(1, \ell) = 1$ and

$$\gcd(s_{i+1}, s_i) = \gcd(\ell s_i - s_{i-1}, s_i) = \gcd(-s_{i-1}, s_i) = \gcd(s_i, s_{i-1}) \,,$$

hence $\gcd(s_{i-1}, s_i) = 1$ for all $i \leq n$.

(ii) Again $\gcd(s_1, s_2) = \gcd(1, \ell) = 1$.

(iii) Here $\gcd(s_2, s_3) = \gcd(\ell, \ell^2 + b) = \gcd(\ell, b)$, which is equal to $-b$ if and only if $b \mid \ell$.

(iv) By Proposition 3.5 this is equivalent to $\gcd(s_i, s_{i-1}) \neq (-b)^{i-2}$ for some i. Let $i = 4$. By definition $s_3 = \ell^2 + b$, $s_4 = \ell^3 + 2\ell b$. Then $\gcd\left(\ell^2 + b, \ell^3 + 2\ell b\right) = \gcd\left(\ell^2 + b, \ell b\right)$. Suppose $\gcd\left(\ell^2 + b, \ell b\right) = b^2$. Then $b^2 \mid \ell b$ and $b^2 \mid \ell^2 + b$. From the first relation we have that $b \mid \ell$, thus $\ell = kb$ for some $k \in \mathbb{Z}$. From the second relation and by substituting ℓ, we have $mb^2 = k^2 b^2 + b$ for some $m \in \mathbb{Z}$. This simplifies to $\left(m - k^2\right) b = -1$. This is a contradiction because $b \neq -1, 1$. $\square$

3.2. **Case 2: $b > 0$.** The key to proving Theorem 3.2 for Case 2 will be to understand exactly how ℓ and b affect $\gcd(s_{n+1}, s_n)$. The assumption throughout this section is that ℓ and b are positive integers.

Definition 3.7. For positive integers ℓ and b, define

$$\begin{aligned} r &= \gcd(\ell, b); \\ t &= \gcd(\ell^2/r, b/r); \\ \sigma &= r/t; \\ \gamma &= \ell/r; \\ \beta &= b/(rt). \end{aligned}$$

Proposition 3.8. *In Definition 3.7, r, t, σ, γ, and β are all positive integers, and*

$$\begin{aligned} \gcd(\ell, b) &= r = \sigma t; \\ \gcd(\ell^2, b) &= rt = \sigma t^2; \\ \ell &= r\gamma = \sigma t\gamma; \\ b &= rt\beta = \sigma t^2 \beta; \\ \gcd(\gamma, \beta) &= 1; \\ \gcd(\gamma, t) &= 1; \\ \gcd(\sigma, \beta) &= 1. \end{aligned}$$

Proposition 3.9. *If $\gcd(\ell, b) = \gcd(\ell^2, b) = r$, then $t = 1$ and $r = \sigma$. In this case,*

$$\begin{aligned} \ell &= r\gamma; \\ b &= r\beta; \\ \gcd(\gamma, \beta) &= 1; \\ \gcd(r, \beta) &= 1. \end{aligned}$$

Lemma 3.10. *For $\mathbf{s}$ defined by (10), assume that ℓ and b are positive integers. For $n \geq 1$, s_n is divisible by $t^{n-1}\sigma^{\lfloor n/2 \rfloor}$.*

Proof. We use induction on n. For the base case, $s_1 = 1$, which is divisible by $t^{1-1}\sigma^{\lfloor 1/2 \rfloor} = 1$; $s_2 = \ell = r\gamma = t\sigma\gamma$, which is divisible by $t^{2-1}\sigma^{\lfloor 2/2 \rfloor} = t\sigma$.

Let $n \geq 3$ and assume the claim true for smaller values. Then by the recurrence for $\mathbf{s}$,

$$s_n = \ell s_{n-1} + b s_{n-2} = t\sigma\gamma s_{n-1} + t^2 \sigma \beta s_{n-2}.$$

If $n = 2k + 1$, then

$$\frac{s_{2k+1}}{t^{2k}\sigma^k} = \frac{t\sigma\gamma s_{2k}}{t^{2k}\sigma^k} + \frac{t^2\sigma\beta s_{2k-1}}{t^{2k}\sigma^k} = \sigma\gamma \frac{s_{2k}}{t^{2k-1}\sigma^k} + \beta \frac{s_{2k-1}}{t^{2k-2}\sigma^{k-1}}. \tag{14}$$

By induction, the two fractions on the right are integers, so s_{2k+1} is divisible by $t^{2k}\sigma^k$.

If $n = 2k$, then

$$\frac{s_{2k}}{t^{2k-1}\sigma^k} = \frac{t\sigma\gamma s_{2k-1}}{t^{2k-1}\sigma^k} + \frac{t^2\sigma\beta s_{2k-2}}{t^{2k-1}\sigma^k} = \gamma \frac{s_{2k-1}}{t^{2k-2}\sigma^{k-1}} + \beta \frac{s_{2k-2}}{t^{2k-3}\sigma^{k-1}}. \tag{15}$$

By induction, the two fractions on the right are integers, so s_{2k} is divisible by $t^{2k-1}\sigma^k$. □

From Lemma 3.10, for $n \geq 1$,

$$\gcd(s_{n+1}, s_n) \quad \text{is divisible by} \quad t^{n-1}\sigma^{\lfloor n/2 \rfloor}.$$

We now show that in the special case $t = 1$, we have equality.

Lemma 3.11. *For* $\mathbf{s}$ *defined by* (10)*, assume that* ℓ *and* b *are positive integers. Let* $\gcd(\ell, b) = \gcd(\ell^2, b) = r$*, then for* $n \geq 1$*,*

$$\gcd(s_{n+1}, s_n) = r^{\lfloor n/2 \rfloor}.$$

Proof. Since $\gcd(\ell, b) = r = \gcd(\ell^2, b)$, we have, from Proposition 3.9,

$$t = 1; \quad \ell = r\gamma; \quad b = r\beta;$$

and

$$\gcd(r, \beta) = 1; \quad \gcd(\beta, \gamma) = 1.$$

We use induction on n. For the base case,

$$\gcd(s_2, s_1) \ = \ \gcd(1, \ell) \ = \ 1 \ = r^{\lfloor 1/2 \rfloor}$$

and

$$\gcd(s_3, s_2) \ = \ \gcd(\ell^2 + b, \ell) \ = \ \gcd(b, \ell) \ = \ r \ = \ r^{\lfloor 2/2 \rfloor}.$$

Let $n \geq 3$ and assume the lemma is true for smaller values. Suppose

$$\gcd(s_{n+1}, s_n) = p\, r^{\lfloor n/2 \rfloor}.$$

We show that $p = 1$.

Assume first that $n = 2k$, so that our hypothesis is

$$\gcd(s_{2k+1}, s_{2k}) = p\, r^k.$$

By the recursion for $\mathbf{s}$,

$$s_{2k+1} \ = \ r\gamma s_{2k} + r\beta s_{2k-1}.$$

Since pr^k divides both s_{2k+1} and s_{2k}, it must also divide $r\beta s_{2k-1}$. By induction, $\gcd(s_{2k}, s_{2k-1}) = r^{k-1}$, so $\frac{s_{2k}}{r^{k-1}}$ and $\frac{s_{2k-1}}{r^{k-1}}$ are relatively prime and therefore pr must divide $r\beta$, i.e., p divides β.

But now, applying the recursion for $\mathbf{s}$ again,

$$s_{2k} \ = \ r\gamma s_{2k-1} + r\beta s_{2k-2}.$$

By Lemma 3.10, s_{2k-2} is divisible by $\sigma^{k-1} = r^{k-1}$. Thus, since p divides β, we have that pr^k divides $r\beta s_{2k-2}$. So also does pr^k divide s_{2k}, and thus pr^k divides $r\gamma s_{2k-1}$. By induction, $\gcd(s_{2k-1}, s_{2k}) = r^{k-1}$, and thus pr is relatively prime to $\frac{s_{2k-1}}{r^{k-1}}$. So it must be that pr divides $r\gamma$, i.e., p divides γ. But now we have p dividing both β and γ, which are relatively prime. So, $p = 1$.

In the case

$$\gcd(s_{2k+2}, s_{2k+1}) = pr^k.$$

By the recursion for $\mathbf{s}$,

$$s_{2k+2} \ = \ r\gamma s_{2k+1} + r\beta s_{2k}.$$

Since pr^k divides both s_{2k+2} and s_{2k+1}, it must also divide $r\beta s_{2k}$. By induction, $\gcd(s_{2k+1}, s_{2k}) = r^k$, so $\frac{s_{2k+1}}{r^k}$ and $\frac{s_{2k}}{r^k}$ are relatively prime and therefore p must divide $r\beta$. Let p' be a prime factor of p.

Suppose first that p' does not divide r. Then p' divides β. Applying the recursion for $\mathbf{s}$ again,

$$s_{2k+1} \;=\; r\gamma s_{2k} + r\beta s_{2k-1}. \tag{16}$$

By Lemma 3.10, s_{2k-1} is divisible by $\sigma^{k-1} = r^{k-1}$. Thus, since p' divides β, we have that $p'r^k$ divides $r\beta s_{2k-1}$. So also does $p'r^k$ divide s_{2k+1}, and thus $p'r^k$ divides $r\gamma s_{2k}$. By induction, $\gcd(s_{2k+1}, s_{2k}) = r^k$, and thus p' is relatively prime to $\frac{s_{2k}}{r^k}$. So it must be that p' divides $r\gamma$. But our assumption was that the prime p' does not divide r, so it must be that p' divides γ. But then we have p' dividing both β and γ, which are relatively prime. So, $p' = 1$.

On the other hand, given that p' divides $r\beta$, if p' divides r, then, since s_{2k} is divisible by r^k, $r\gamma s_{2k}$ is divisible by $p'r^k$. So also does $p'r^k$ divide s_{2k+1}, and thus, by (16), $p'r^k$ divides $r\beta s_{2k-1}$. But since $\gcd(s_{2k}, s_{2k-1}) = r^{k-1}$, $p'r$ is relatively prime to s_{2k-1}/r^{k-1} and therefore $p'r$ divides $r\beta$, so p' divides β. But $\gcd(r, \beta) = 1$, so $p' = 1$. Thus $p = 1$. □

By Lemma 3.10, for arbitrary positive integers ℓ and b, $\gcd(s_{n+1}, s_n)$ is divisible by $t^{n-1}\sigma^{\lfloor n/2 \rfloor}$. However, in contrast to the case when $t = 1$, equality does not always hold. For example, if $\ell = 6$ and $b = 36$, then $r = 6$, $t = 6$, and $\sigma = 1$, so $t^{n-1}\sigma^{\lfloor n/2 \rfloor} = 6^{n-1}$ and we get the following:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\frac{\gcd(s_{n+1},s_n)}{6^{n-1}}$	1	1	2	3	1	2	1	3	2	1	1	6	1	1	2	3	1	2	1	3	2	1	1	6

As another example, if $\ell = 6 \cdot 3 \cdot 5$ and $b = 36 \cdot 3 \cdot 7$, then $r = 6 \cdot 3$, $t = 6$, and $\sigma = 3$, so $t^{n-1}\sigma^{\lfloor n/2 \rfloor} = 6^{n-1}3^{\lfloor n/2 \rfloor}$ and we get the following:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\frac{\gcd(s_{n+1},s_n)}{6^{n-1}3^{\lfloor n/2 \rfloor}}$	1	1	2	1	1	6	1	1	2	1	1	6	1	1	2	1	1	6	1	1	2	1	1	6

What we *will* be able to show is that $\frac{\gcd(s_{n+1},s_n)}{t^{n-1}\sigma^{\lfloor n/2 \rfloor}}$ is a factor of t, something that might be conjectured from the evidence in the tables. In order to prove this, we need another fact.

Lemma 3.12. *For* $\mathbf{s}$ *defined by* (10)*, assume that* ℓ *and* b *are positive integers. Then for* $n \geq 0$*,*

$$s_n = t^{n-1} f_n,$$

where f *is defined by the recurrence*

$$f_n \;=\; \tfrac{\ell}{t} f_{n-1} + \tfrac{b}{t^2} f_{n-2},$$

with initial conditions $f_0 = 0$, $f_1 = 1$*. Furthermore,* f*, so defined, is an integer sequence satisfying*

$$\gcd(f_{n+1}, f_n) \;=\; \sigma^{\lfloor n/2 \rfloor}.$$

Proof. First, to show that $s_n = t^{n-1} f_n$, we use induction. For the base case, $s_0 = 0$, $s_1 = 1$, so the claim is true. Let $n \geq 1$ and assume it is true for values less than or equal to n. Then

$$s_{n+1} \;=\; \ell s_n + b s_{n-1} \;=\; \ell t^{n-1} f_n + b t^{n-2} f_{n-1} \;=\; t^n \left(\tfrac{\ell}{t} f_n + \tfrac{b}{t^2} f_{n-1} \right) \;=\; t^n f_{n+1}.$$

To see that f is an integer sequence, from Proposition 3.8, $\frac{\ell}{t} = \sigma\gamma$ and $\frac{b}{t^2} = \sigma\beta$, and σ, γ, and β are positive integers. Finally, to prove the assertion about $\gcd(f_{n+1}, f_n)$, observe that

$$\gcd\left(\tfrac{\ell}{t}, \tfrac{b}{t^2}\right) \;=\; \gcd(\sigma\gamma, \sigma\beta) \;=\; \sigma,$$

since γ and β are relatively prime. Furthermore,

$$\gcd\left(\left(\tfrac{\ell}{t}\right)^2, \tfrac{b}{t^2}\right) \;=\; \gcd\left(\sigma^2\gamma^2, \sigma\beta\right) \;=\; \sigma,$$

since β is relatively prime to both γ and σ.

Since $\gcd\left(\frac{\ell}{t}, \frac{b}{t^2}\right) = \gcd\left(\left(\frac{\ell}{t}\right)^2, \frac{b}{t^2}\right)$, the sequence f satisfies the hypothesis of Lemma 3.11 and therefore the final claim follows. $\square$

We can now show that

$$t^{n-1}\sigma^{\lfloor n/2 \rfloor} \;\Big|\; \gcd(s_{n+1}, s_n) \;\Big|\; t^n \sigma^{\lfloor n/2 \rfloor}.$$

Lemma 3.13. *For* $\mathbf{s}$ *defined by* (10)*, assume that* ℓ *and* b *are positive integers. For* $n \geq 1$*,* $\gcd(s_{n+1}, s_n)$ *divides* $t^n \sigma^{\lfloor n/2 \rfloor}$*.*

Proof. Using Lemma 3.12,

$$\begin{aligned} \gcd(s_{n+1}, s_n) \;&=\; \gcd(t^n f_{n+1}, t^{n-1} f_n) \;=\; t^{n-1} \gcd(t f_{n+1}, f_n) \\ &=\; t^{n-1}\sigma^{\lfloor n/2 \rfloor} \gcd\left(t\, \frac{f_{n+1}}{\sigma^{\lfloor n/2 \rfloor}}, \frac{f_n}{\sigma^{\lfloor n/2 \rfloor}} \right). \end{aligned}$$

By Lemma 3.12, the fractions in the last line are relatively prime integers. Thus,

$$\gcd(s_{n+1}, s_n) \;=\; t^{n-1}\sigma^{\lfloor n/2 \rfloor} \gcd\left(t, \frac{f_n}{\sigma^{\lfloor n/2 \rfloor}} \right),$$

which is a divisor of $t^{n-1}\sigma^{\lfloor n/2 \rfloor} t = t^n \sigma^{\lfloor n/2 \rfloor}$. $\square$

We need one more lemma to prove Theorem 3.2, implying that the sequence $\frac{s_n}{\gcd(s_n, s_{n+1})}$ grows without bound.

Lemma 3.14. *For* $n \geq 1$*, define* h_n *by*

$$h_n \;=\; \frac{s_n}{t^{n-1}\sigma^{\lfloor n/2 \rfloor}}.$$

The sequence h_n *satisfies the recurrence*

$$\begin{aligned} h_{2k+1} &= \sigma\gamma h_{2k} + \beta h_{2k-1} \\ h_{2k} &= \gamma h_{2k-1} + \beta h_{2k-2} \end{aligned}$$

with initial conditions $h_0 = 0$*,* $h_1 = 1$*.*

Proof. We go by induction. Using $s_1 = 1$, $s_2 = \ell$, and $s_3 = \ell^2 + b$, for $k = 1$,

$$h_2 \;=\; \frac{s_2}{t\sigma} \;=\; \frac{\ell}{t\sigma} \;=\; \frac{t\sigma\gamma}{t\sigma} \;=\; \gamma \;=\; \gamma h_1 + \beta h_0;$$

$$h_3 \;=\; \frac{s_3}{t^2\sigma} \;=\; \frac{\ell^2 + b}{t^2\sigma} \;=\; \frac{t^2\sigma^2\gamma^2 + t^2\sigma\beta}{t^2\sigma} \;=\; \sigma\gamma^2 + \beta \;= \sigma\gamma h_2 + \beta h_1.$$

Let $k \geq 2$ and assume the claim is true for smaller values. Then, making use of (14) and (15),

$$h_{2k+1} \;=\; \frac{s_{2k+1}}{t^{2k}\sigma^k} \;=\; \sigma\gamma\frac{s_{2k}}{t^{2k-1}\sigma^k} + \beta\frac{s_{2k-1}}{t^{2k-2}\sigma^{k-1}} \;=\; \sigma\gamma h_{2k} + \beta h_{2k-1}\,,$$

and

$$h_{2k} \;=\; \frac{s_{2k}}{t^{2k-1}\sigma^k} \;=\; \gamma\frac{s_{2k-1}}{t^{2k-2}\sigma^{k-1}} + \beta\frac{s_{2k-2}}{t^{2k-3}\sigma^{k-1}} \;=\; \gamma h_{2k-1} + \beta h_{2k}\,. \qquad \square$$

Since σ, γ, β are all positive integers, the terms in the sequence h_n go to infinity. In particular:

Corollary 3.15. *For h_n defined by*

$$h_n \;=\; \frac{s_n}{t^{n-1}\sigma^{\lfloor n/2\rfloor}}\,,$$

we have that $h_n \geq F_n$, where F_n is the n-th Fibonacci number.

We can now complete the proof of Theorem 3.2, which follows from Theorem 3.6 (the $b < 0$ case) and Theorem 3.16 (the $b > 0$ case).

Theorem 3.16. *Let ℓ and b be positive integers and let* $\mathbf{s}$ *be defined by* (10)*. Choose k large enough so that*

$$\frac{s_{2k+1}}{t^{2k}\sigma^k} > t(t+b)\,,$$

where t and σ are as in Definition 3.7. Then the n-dimensional $\mathbf{s}$*-lecture hall cone $\mathcal{C}_n^{(\mathbf{s})}$ is not Gorenstein for $n \geq 2k+2$.*

Proof. Note that by Corollary 3.15, such choice of k is always possible.

By Lemma 3.13, there are positive integers u and v such that $u|t$ and $v|t$ and such that

$$\gcd(s_{2k+2}, s_{2k+1}) \;= ut^{2k}\sigma^k; \tag{17}$$

$$\gcd(s_{2k+1}, s_{2k}) \;= vt^{2k-1}\sigma^k. \tag{18}$$

If $\mathcal{C}_{2k+2}$ is Gorenstein, then by Corollary 2.4 the Gorenstein point $\mathbf{c}$ satisfies

$$c_{2k+1}s_{2k} \;= c_{2k}s_{2k+1} + vt^{2k-1}\sigma^k \tag{19}$$

$$c_{2k+2}s_{2k+1} \;= c_{2k+1}s_{2k+2} + ut^{2k}\sigma^k, \tag{20}$$

and all coodinates of $\mathbf{c}$ are integers. Rewriting (19),

$$c_{2k+1}\,\frac{s_{2k}}{vt^{2k-1}\sigma^k} \;-\; c_{2k}\,\frac{s_{2k+1}}{vt^{2k-1}\sigma^k} \;=\; 1\,.$$

Note that by (18), the fractions are integers and therefore the integers c_{2k+1} and $\frac{s_{2k+1}}{vt^{2k-1}\sigma^k}$ are relatively prime. Combining (19) and (20),

$$ut(c_{2k+1}s_{2k} - c_{2k}s_{2k+1}) \;=\; uvt^{2k}\sigma^k \;=\; v(c_{2k+2}s_{2k+1} - c_{2k+1}s_{2k+2}),$$

so

$$c_{2k+1}(uts_{2k} + vs_{2k+2}) \;=\; s_{2k+1}(vc_{2k+2} + utc_{2k}).$$

Then

$$c_{2k+1}\left(\frac{uts_{2k}}{vt^{2k-1}\sigma^k} + \frac{vs_{2k+2}}{vt^{2k-1}\sigma^k}\right) \;=\; \frac{s_{2k+1}}{vt^{2k-1}\sigma^k}\left(vc_{2k+2} + utc_{2k}\right).$$

Again, by (17) and (18), all the fractions here are integers. We now use the fact that c_{2k+1} and $\frac{s_{2k+1}}{vt^{2k-1}\sigma^k}$ are relatively prime to conclude that

$$\frac{s_{2k+1}}{vt^{2k-1}\sigma^k} \quad \text{divides} \quad \frac{uts_{2k}}{vt^{2k-1}\sigma^k} + \frac{vs_{2k+2}}{vt^{2k-1}\sigma^k}.$$

Multiplying through by the denominator, we get that

$$s_{2k+1} \quad \text{divides} \quad uts_{2k} + vs_{2k+2} \;=\; uts_{2k} + v\ell s_{2k+1} + vbs_{2k},$$

where in the last step we have applied the recursion for $\mathbf{s}$. But now this means that

$$s_{2k+1} \quad \text{divides} \quad uts_{2k} + vbs_{2k} \;=\; s_{2k}(ut + vb).$$

Since

$$\gcd(s_{2k+1}, s_{2k}) \;=\; vt^{2k-1}\sigma^k,$$

the integers $\frac{s_{2k+1}}{(vt^{2k-1}\sigma^k)}$ and s_{2k} have no common factors and therefore

$$\frac{s_{2k+1}}{vt^{2k-1}\sigma^k} \quad \text{divides} \quad ut + vb.$$

In particular, it must be that

$$\frac{s_{2k+1}}{vt^{2k-1}\sigma^k} \;\leq\; ut + vb.$$

But by choice of k,

$$\frac{s_{2k+1}}{t^{2k}\sigma^k} \;>\; t(t+b),$$

which implies

$$\frac{s_{2k+1}}{vt^{2k-1}\sigma^k} \;>\; \frac{t^2(t+b)}{v} \;\geq\; t(t+b),$$

since $v \leq t$. However, since $u|t$ and $v|t$,

$$ut + vb \;\leq\; t^2 + tb,$$

a contradiction. Thus $\mathcal{C}^{(\mathbf{s})}_{2k+1}$ is not Gorenstein and, therefore, by Corollary 2.5, $\mathcal{C}_n$ is not Gorenstein for $n \geq 2k+2$. □

Corollary 3.17. *If* $\gcd(\ell, b) = \gcd(\ell^2, b)$ *then* $\mathcal{C}_n$ *is not Gorenstein for* $n \geq 5$.

Proof. By the previous proof and Corollary 2.5, we need only check that $\frac{s_5}{t^4\sigma^2} > t(t+b)$. If $\gcd(\ell, b) = \gcd(\ell^2, b)$ then $t = 1$ and $\sigma = r = \gcd(\ell, b)$. So, we check that $s_5 > r^2 + r^2 b$. From the recursive definition of $\mathbf{s}$, we have $s_5 = \ell^4 + 3b\ell^2 + b^2$ and

$$\ell^4 + 3b\ell^2 + b^2 \;\geq r^4 + 3r^2 b + r^2 \;>\; r^2 + r^2 b.$$

□

One can show additionally that when $\gcd(\ell, b) = \gcd(\ell^2, b)$, $\mathcal{C}_1^{(\mathbf{s})}$ and $\mathcal{C}_2^{(\mathbf{s})}$ are Gorenstein; that $\mathcal{C}_3^{(\mathbf{s})}$ is Gorenstein if and only if $b + r$ is a multiple of ℓ; that $\mathcal{C}_4^{(\mathbf{s})}$ is Gorenstein if and only if both $b + r$ is a multiple of ℓ and $\ell^2 + b$ divides $r(b + 1)$. This last situation happens, for example, when $\ell = 1$, or when $\ell = b$.

4. Ehrhart theory and symmetric h^*-vectors

Another consequence of our main result involves the h^*-vector of the polytope $\mathcal{R}_n^{(\mathbf{s})}$ associated with the $\mathbf{s}$-lecture hall partitions, introduced in Section 1.3.

Let

$$i\left(\mathcal{R}_n^{(\mathbf{s})}, t\right) \; = \left|\left\{\lambda \in \mathcal{L}_n^{(\mathbf{s})} \,:\, \lambda_n \leq t\right\}\right|.$$

The function $i\left(\mathcal{R}_n^{(\mathbf{s})}, t\right)$ is known to be a *quasi-polynomial* in t, i.e., it has the form $a_n(t)\, t^n + a_{n-1}(t)t^{n-1} + \cdots + a_0(t)$, where $a_0(t), a_1(t), \ldots, a_n(t)$ are periodic functions of t; the lcm of their periods is the *period* of $\mathcal{R}_n^{(\mathbf{s})}$ [1, 6]. The *Ehrhart series* of $\mathcal{R}_n^{(\mathbf{s})}$ is the series

$$\mathcal{E}_n^{(\mathbf{s})}(x) = \sum_{t \geq 0} i\left(\mathcal{R}_n^{(\mathbf{s})}, t\right) x^t.$$

It was shown in [8] that

$$\mathcal{E}_n^{(\mathbf{s})}(x) \; = \; \frac{Q_n^{(\mathbf{s})}(x)}{(1 - x^{s_n})^{n+1}}\,,$$

where $Q_n^{(\mathbf{s})}(x)$ is a polynomial with positive integer coefficients which can be interpreted in terms of statistics on "$\mathbf{s}$-inversion sequences". Furthermore, $Q_n^{(\mathbf{s})}(1) = s_1 s_2 \cdots s_n^2$. The coefficient sequence of $Q_n^{(\mathbf{s})}(x)$ is referred to as the h^*-vector of the polytope $\mathcal{R}_n^{(\mathbf{s})}$. For example,

$$\mathcal{E}_n^{(1,3,5)}(x) = \frac{1 + 2x + 4x^2 + 6x^3 + 9x^4 + 10x^5 + 11x^6 + 10x^7 + 9x^8 + 6x^9 + 4x^{10} + 2x^{11} + x^{12}}{(1 - x^5)^4}\,,$$

so the h^*-vector of the polytope $\mathcal{R}_n^{(1,3,5)}$ is $[1, 2, 4, 6, 9, 10, 11, 10, 9, 6, 4, 2, 1]$. (Actually, $Q_n^{(\mathbf{s})}(x)$ and $(1 - x^{s_n})^{n+1}$ always have common factors [8, Corollary 3], but we won't make use of this here.)

Which sequences $\mathbf{s}$ give rise to symmetric h^*-vectors? Conjectures were made in [8] which we will resolve here.

Theorem 4.1. *For a sequence $\mathbf{s}$ of positive integers, the h^*-vector of $\mathcal{R}_n^{(\mathbf{s})}$ is symmetric if and only if $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein.*

Proof. Note that the h^*-vector of $\mathcal{R}_n^{(\mathbf{s})}$ is symmetric if and only if $Q_n^{(\mathbf{s})}(x)$ is self-reciprocal, which occurs if and only if $\mathcal{E}_n^{(\mathbf{s})}(x)$ is self-reciprocal.

The weight function $w(\lambda) = \lambda_n$ satisfies the conditions of Theorem 2.1, so $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if the function $f(x)$, defined by

$$f(x) = \sum_{\lambda \in \mathcal{L}_n^{(\mathbf{s})}} x^{\lambda_n},$$

is self reciprocal. To relate $f(x)$ to $\mathcal{E}_n^{(\mathbf{s})}(x)$,

$$\begin{aligned} f(x) &= \sum_{\lambda\in\mathcal{L}_n^{(\mathbf{s})}} x^{\lambda_n} = \sum_{t\geq 0} \sum_{\lambda\in\mathcal{L}_n^{(\mathbf{s})};\,\lambda_n=t} x^t \\ &= 1 + \sum_{t\geq 1} x^t \left(i\left(\mathcal{R}_n^{(\mathbf{s})}, t\right) - i\left(\mathcal{R}_n^{(\mathbf{s})}, t-1\right)\right) \\ &= (1-x)\sum_{t\geq 0} x^t i\left(\mathcal{R}_n^{(\mathbf{s})}, t\right) \end{aligned}$$

Thus, $\mathcal{E}_n^{(\mathbf{s})}(x) = f(x)/(1-x)$, which is self-reciprocal if and only if $f(x)$ is. □

Remark 4.2. For the geometrically inclined, Theorem 4.1 also follows from the fact that $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if the cone over $\mathcal{R}_n^{(\mathbf{s})}$ is Gorenstein (see [1, Section 3.1] for the definition of a cone over a polytope). This follows easily from the observation that if $(c_1, \ldots, c_n)$ is the Gorenstein point for $\mathcal{C}_n^{(\mathbf{s})}$, then $(c_1, \ldots, c_n, c_n + 1)$ is the Gorenstein point for the cone over $\mathcal{R}_n^{(\mathbf{s})}$.

Combining Theorems 4.1 and 2.7, we have:

Theorem 4.3. *Let* $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ *be a sequence of positive integers such that* $\gcd(s_i, s_{i+1}) = 1$ *for* $1 \leq i < n$. *Then the* h^**-vector of* $\mathcal{R}_n^{(\mathbf{s})}$ *is symmetric if and only if* $\mathbf{s}$ *is* $\mathbf{u}$*-generated by a sequence of positive integers.*

For example, in Section 2.3, the "1 mod k" sequences are $\mathbf{u}$-generated by $\mathbf{u} = (k+2, 2, 2, 2, \ldots)$, so by Theorem 4.3, the h^*-vector of $\mathcal{R}_n^{(\mathbf{s})}$ is symmetric, confirming [8, Conjecture 5.5].

Combining Theorems 4.1 and 3.2, we have the following.

Theorem 4.4. *Let* b *and* ℓ *be integers satisfying* $0 < |b| \leq \ell$. *Let* $\mathbf{s}$ *be defined by the recurrence*

$$s_n = \ell s_{n-1} + b s_{n-2},$$

with initial conditions $s_1 = 1$, $s_0 = 1$. *The* h^**-vector of* $\mathcal{R}_n^{(\mathbf{s})}$ *is symmetric for all* $n \geq 0$ *if and only if* $b = -1$. *If* $b \neq -1$, *there is an integer* $n_0 = n_0(b, \ell)$ *so that for all* $n \geq n_0$, *the* h^**-vector of* $\mathcal{R}_n^{(\mathbf{s})}$ *is not symmetric.*

The first conclusion in Theorem 4.4 proves [8, Conjectures 5.6 and 5.7].

5. Concluding remarks

We summarize the main contributions of this paper and offer two related problems.

For partition theory, our main contibution is the negative result that among the sequences $\mathbf{s}$ defined by $s_n = \ell s_{n-1} + b s_{n-1}$ with initial conditions $s_0 = 0, s_1 = 1$, if $\mathbf{s}$ is not an ℓ-sequence, the generating function for the $\mathbf{s}$-lecture hall partitions can be of the form $((1-q^{e_1})\cdots(1-q^{e_n}))^{-1}$ for at most finitely many n.

We proved a geometric result showing that ℓ-sequences are unique among sequences defined by second-order linear recurrences. Before this, we knew special things about ℓ-sequences: they generalized the integers ($\ell = 2$); they give rise to an ℓ-generalization of Euler's Partition Theorem [4, 11]; they give rise to an ℓ-nomial coefficient which has a q-analogue with a meaningful interpretation

[7]. But until now, we did not have negative results to distinguish ℓ-sequences from other second order linear recurrences.

The theory of Gorenstein cones provided a direct translation of results about lecture hall partitions into results about lecture hall cones. Our results provide many new examples of Gorenstein cones, in particular, the **s**-lecture hall cones where $\gcd(s_{i+1}, s_1) = 1$ and **s** is **u**-generated. In Ehrhart theory, we proved that the h^*-vector of the rational **s**-lecture hall polytope is symmetric when **s** is an ℓ-sequence. And, except for finitely many n, it is not for any other second order sequence.

For ℓ-sequences, the shift vector in the Gorenstein condition is

$$(s_1, s_1 + s_2, s_2 + s_3, \ldots, s_{n-1} + s_n)$$

while the generating function for s-lecture hall partitions is

$$\sum_{\lambda \in L_n^{(s)}} q^{|\lambda|} = \frac{1}{(1-q^{s_1})(1-q^{s_1+s_2})\cdots(1-q^{s_{n-1}+s_n})}.$$

Can we explain why the entries in the shift vector and the exponents are the same? Is there an explanation at all? Or is this just coincidence?

Is it true that for every sequence **s** of positive integers, the h^*-vector of the rational s-lecture hall polytope is unimodal (as, e.g., for the "1 mod k" sequences)? We have tested many sequences **s**, including ones that were not themselves monotone and found no counterexamples. The polynomials $Q_n^{(\mathbf{s})}(x)$ do not, in general, have all roots real.

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